AIAA JOURNAL
Vol. 31, No. 2, February 1993

Vibration of a Sandwich Plate Strip of Linearly Varying Thickness

A. P. Gupta* and R. R. Bhargava†
University of Roorkee, Roorkee 247667, India

Free transverse vibration of a sandwich plate strip having a core of linearly varying thickness and symmetrical faces of constant thickness is considered. The materials of the core and the faces are assumed to be linearly elastic and isotropic. The effects of transverse shear deformation and rotary inertia are included in the core as well as in the faces. The equations of motion derived by Hamilton's energy principle are solved by the Chebyshev collocation method. Frequencies and mode shapes for the first four normal modes of vibration are plotted in figures by varying the taper angle, thickness of the faces, initial thickness of the core, and ratio of the Young's moduli of the faces and the core. Frequencies are also plotted by varying the taper angle and ratio of the densities of the faces to the core in such a way that the total weight of the plate remains constant.

Introduction

In recent years, sandwich constructions have gained a well-established position due to their applications in aerospace and marine industries. Sandwich constructions are lighter in weight and have very high rigidity. The structural elements of sandwich construction may be of varying thickness for further reduction in weight or due to some specific design requirements. Sandwich plates of varying thickness are being used for trailing edges of airplane lifting surfaces.

Habip,[1] Plantema,[2] and Bert[3] have given reviews of the work on sandwich constructions available in literature until 1979. Almost all of the work available to date on sandwich structural elements is confined to elements of constant thickness. A few papers available after 1979 on sandwich beams and plates of varying thickness are as follows: Gupta and Jain[4] and Paydar and Adams[5] have considered the vibration of annular sandwich plates of linearly varying thickness by taking faces as membranes. Taber and Viano[6] have compared analytical and experimental results for free vibration of nonuniform composite beams. Paydar and Libove[7,8] and Paydar[9] have considered the static behavior of sandwich plates of varying thickness. Ko[10] has analyzed free vibration of rotating sandwich tapered beams. Kar and Sujata[11] have considered the dynamic stability of tapered symmetric sandwich beams.

In the present paper, free transverse vibration of a sandwich plate strip of infinite length, finite breadth, and linearly varying thickness along the breadth is considered. The faces of the plate are taken of constant thickness. They are taken identical in material and dimensions and equally inclined to the middle plane of the plate. The core of the plate is taken of linearly varying thickness. Apart from flexural rigidity, transverse shear and rotatory inertia of the core as well as of the faces are included in the analysis. No restriction is imposed on magnitudes of the ratios between the thickness, material densities, and elastic constants of the core and the faces except that they are isotropic and linearly elastic. The equations of motion and edge conditions are derived by Hamilton's energy principle. They are solved by the Chebyshev collocation method. Frequencies for the first four normal modes of vibration are computed for clamped and cantilever plates for various values of the taper angle, thicknesses of the faces and the core, and the ratio of the Young's moduli of the faces and the core. Frequencies are also computed by varying the taper angle and ratio of the densities of the faces and the core in such a way that the total weight of the plate remains constant. Normalized transverse deflections are also computed for the first four normal modes of vibration. To check the validity of the theory and the numerical method, frequencies are compared with monocoque plates of linearly varying thickness and analytic solution of a sandwich plate of constant thickness. A close agreement is found.

Received Jan. 13, 1992; revision received July 22, 1992; accepted for publication July 27, 1992.

*Professor, Department of Mathematics.
†Reader, Department of Mathematics.

Formulation of the Problem

A tapered sandwich plate of infinite length and finite breadth a is considered. The core of the plate is taken of thickness $2h_1$, varying linearly along the breadth. Each of the two faces of the plate are taken of constant thickness h_f. The plate is referred to rectangular coordinates by taking the long sides of the plate in the planes $x = 0$ and a, and extending to infinity in both of the directions of y axis. The middle plane, lower and upper interfaces, and bottom and top of the plate are taken as $z = 0$, $\mp h$, and $\mp (h_1 + h_2)$, respectively, where α is the taper angle of the core thickness and $h_2 = h_f \sec \alpha$. A section of the plate along the zx plane is shown in Fig. 1. Both of the faces of the plate are taken of the same material and dimensions but different from that of the core. The various quantities for the core, the lower face, and the upper face will be distinguished by the subscripts 1, 2, and 3, respectively.

The displacement components v_i, w_i, and u_i, $i = 1, 2, 3$, in the directions of y, z, and x axes, respectively, are approximated according to Yu[12] as

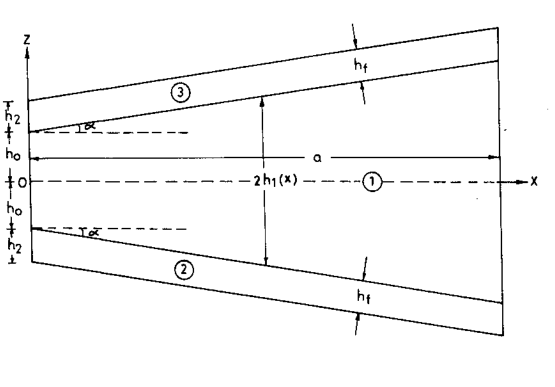

Fig. 1 Section of the plate along the z plane.

$$v_i(x,z,t)=0, \quad w_i(x,z,t)=w(x,t), \quad u_1(x,z,t)=z\beta_1(x,t)$$
$$\begin{matrix} u_2(x,z,t) \\ u_3(x,z,t) \end{matrix} = \mp h_1(x)\beta_1(x,t)+[z\pm h_1(x)]\beta_2(x,t) \tag{1}$$

where β_1 and β_2 are angles of rotation in the zx plane of the lines of the core and the faces originally normal to the middle plane of the plate.

The nonzero strain components are found to be

$$\epsilon_{x1}=z\beta_{1,x}, \quad \epsilon_{zx1}=\beta_1+w_{,x}, \quad \epsilon_{zx2}=\epsilon_{zx3}=\beta_2+\omega_{,x}$$
$$\begin{matrix} \epsilon_{x2} \\ \epsilon_{x3} \end{matrix} = \mp h_1\beta_{1,x}+(z\pm h_1)\beta_{2,x}\mp h_{1,x}(\beta_1-\beta_2) \tag{2}$$

where a comma followed by a suffixed variable denotes differentiation with respect to that variable.

The equations of motion derived by Hamilton's energy principle are

$$m_{x,x}-p_{x1}-2h_1^2(\rho_1h_1/3+\rho_2h_2)\beta_{1,tt}-\rho_2h_1h_2^2\beta_{2,tt}=0 \tag{3a}$$
$$n_{x,x}-p_{x2}-\rho_2h_2^2(h_1\beta_{1,tt}+2h_2\beta_{2,tt}/3)=0 \tag{3b}$$
$$q_{x,x}-2(\rho_1h_1+\rho_2h_2)w_{,tt}=0 \tag{3c}$$

and the edge conditions are vanishing of one member of each of the following pairs:

$$(m_x,\beta_1), \quad (n_x,\beta_2), \quad (q_x,w)$$

where ρ_1 and ρ_2 are the densities of the core and the faces respectively, and

$$m_x=m_{x1}h_1(n_{x3}-n_{x2}), \quad n_x=m_{x2}+m_{x3}-h_1(n_{x3}-n_{x2})$$
$$q_x=q_{x1}+q_{x2}+q_{x3}, \quad p_{x1}=q_{x1}+h_{1,x}(n_{x3}-n_{x2}) \tag{4}$$
$$p_{x2}=q_{x2}+q_{x3}-h_{1,x}(n_{x3}-n_{x2})$$

$$(n_{xi},m_{xi},q_{xi})=\int(\sigma_{xi},z\sigma_{xi},K\sigma_{zxi})\,dz \tag{5}$$

K is the average shear constant. The limits of integration for the suffix $i=1$, 2, and 3 are $-h_1$ to h_2, $-(h_1+h_2)$ to $-h_1$, and h_1 to (h_1+h_2), respectively.

Since the materials of the core and the faces are linearly elastic and isotropic, the stress-strain relations are taken as

$$\sigma_{xi}=\lambda_i\epsilon_{xi}, \quad \sigma_{xzi}=\mu_i\epsilon_{xzi} \tag{6a}$$
$$\lambda_i=E_i/(1-\nu_i^2), \quad \mu_i=E_i/[2(1+\nu_i)] \tag{6b}$$

where E_i and ν_i are Young's moduli and Poisson's ratios, respectively.

For harmonic vibrations, β_1, β_2, and w are taken as

$$[\beta_1(x,t),\beta_2(x,t),w(x,t)]=[B_1(x),B_2(x),aW(x)]e^{i\omega t} \tag{7}$$

where w is the circular frequency of vibration.

With the help of Eqs. (2) and (4–7), the equations of motion (3) obtained in terms of displacements are reduced to nondimensional form:

$$A_0B_{1,XX}+A_1B_{1,X}+A_2B_1+A_3B_{2,XX}+A_4B_{2,X}+A_5B_2+A_6W_{,X}=0 \tag{8a}$$
$$A_7B_{1,XX}+A_8B_{1,X}+A_9B_1+A_{10}B_{2,XX}+A_{11}B_2+A_{12}W_{,X}=0 \tag{8b}$$
$$A_{13}B_{1,X}+A_{14}B_1+A_{15}B_{2,X}+A_{16}W_{,XX}+A_{17}W_{,X}+A_{18}W=0 \tag{8c}$$

where

$$A_0=8H_1(R_1H_1+3R_2H_2)$$
$$A_1=24(R_1H_1+2R_2H_2)H_{1,X} \tag{9a}$$
$$A_2=2H_1(H_1+3R_\rho H_2)\Omega^2-6K, \quad A_3=12R_2H_2^2$$
$$A_4=-24R_2H_2H_{1,X} \tag{9b}$$
$$A_5=3R_\rho H_2^2\Omega^2, \quad A_6=-12K, \quad A_7=12R_2H_1H_2$$
$$A_8=24R_2(H_1+H_2)H_{1,X} \tag{9c}$$
$$A_9=24R_2H_{1,X}^2+3R\rho H_1H_2\Omega^2, \quad A_{10}=2A_3/3$$
$$A_{11}=2A_5/3-24R_2H_{1,X}^2-6R_\mu K \tag{9d}$$
$$A_{12}=-12R_\mu K, \quad A_{13}=2H_1K, \quad A_{14}=2H_{1,X}K$$
$$A_{15}=2R_\mu H_2K, \quad A_{16}=2(A_{13}+A_{15}) \tag{9e}$$
$$A_{17}=2A_{14}, \quad A_{18}=(H_1+R_\rho H_2)\Omega^2$$
$$\Omega^2=\rho_1a^2\omega^2/\mu_1 \tag{9f}$$

$$X=(2x-a)/a, \quad H_1=h_1/a, \quad H_2=h_2/a \tag{10a}$$
$$R_1=2/(1-\nu_1), \quad R_2=2(1+\nu_1)R_e/(1-\nu_1^2R_\nu^2)$$
$$R_\mu=(1+\nu_1)R_e/(1+\nu_1R_\nu) \tag{10b}$$
$$R_\nu=\nu_2/\nu_1, \quad R_e=E_2/E_1, \quad R_\rho=\rho_2/\rho_1 \tag{10c}$$

In Eqs. (9) the above expression, Ω is the nondimensional frequency parameter.

The linearly varying thickness of the core is taken as

$$H_1=H_0+(X+1)\tan\alpha/2 \tag{11}$$

where $H_0=h_0/a$, and h_0 is the thickness of the core at $x=0$.

Nondimensional bending moments and shear forces are given by

$$M_X=m_x/(a^2\mu_1)=(H_1/6)[A_0B_{1,X}+A_3B_{2,X}-A_4(B_1-B_2)]e^{i\omega t} \tag{12a}$$
$$N_X=n_x/(a^2\mu_1)=(A_3/18)[3H_1B_{1,X}+2H_2B_{2,X}+3H_{1,X}(B_1-B_2)]e^{i\omega t} \tag{12b}$$
$$Q_X=q_x/(a\mu_1)=(A_{13}B_1+A_{15}B_2+A_{16}W_{,X})e^{i\omega t} \tag{12c}$$

Solution by the Chebyshev Collocation Method

According to this method, one takes

$$(B_{1,XX},B_{2,XX},W_{,XX})=\Sigma(a_j,b_j,c_j)T_{j-3} \tag{13}$$

where summation over j is taken from 3 to n; a_j, b_j, and c_j are unknown constants; and T_j are Chebyshev polynomials defined as

$$T_0=1, \quad T_1=X, \quad T_j=2XT_{j-1}-T_{j-2}, \quad j>2 \tag{14}$$

Integration of Eq. (13) gives

$$B_{1,X}=a_2+\Sigma a_jT_{j-3}^1, \quad B_1=a_1+a_2T_1+\Sigma a_jT_{j-3}^2 \tag{15a}$$

Table 1 Ω for sandwich plate of constant thickness

Mode of vibration	C-C plate $H_0=0.04$, $H_f=0.01$ CCM	AS	$H_0=0.02$, $H_f=0.01$ CCM	AS	F-C plate $H_0=0.04$, $H_f=0.01$ CCM	AS	$H_0=0.02$, $H_f=0.03$ CCM	AS
Fundamental	1.52153	1.52153	2.57930	2.57930	0.63556	0.63556	0.62570	0.62570
Second 3.33621	3.33621	6.83985	6.83985	2.05175	2.05175	2.76750		2.76750
Third 5.63808	5.63808	13.09184	13.09184	3.87444	3.87444	6.99922		6.99922
Fourth 8.50521	21.23460	8.50521	6.12644	21.23460	6.12644	13.23150		13.23150

$$B_{2,X}=b_2+\Sigma b_j T^1_{j-3},\qquad B_2=b_1+b_2T_1+\Sigma b_j T^2_{j-3} \tag{15b}$$

$$W_{,X}=c_2+\Sigma c_j T^1_{j-3},\qquad W=c_1+c_2T_1+\Sigma c_j T^2_{j-3} \tag{15c}$$

where

$$T_j^{k+1}=\int T_j^k\,\mathrm{d}x=\tfrac{1}{2}[T^k_{j+1}/(j+1)-T^k_{j-1}/(j-1)],\qquad j>1 \tag{16a}$$

$$T_0^{k+1}=T_1^k,\qquad T_1^{k+1}=(T_2^k+T_0^k)/4$$
$$k=0,1,\qquad T_j^0=T_j \tag{16b}$$

Substitution of B_1, B_2, W, and their derivatives given by Eqs. (13) and (15) into Eqs. (8) gives

$$A_2a_1+(A_1+A_2T_1)a_2+\Sigma(A_0T_{j-3}+A_1T^1_{j-3}+A_2T^2_{j-3})a_j$$
$$+A_5b_1+(A_4+A_5T_1)b_2+\Sigma(A_3T_{j-3}+A_4T^1_{j-3}+A_5T^2_{j-3})$$
$$\times b_j+A_6c_2+\Sigma A_6T^1_{j-3}c_j=0 \tag{17a}$$

$$A_9a_1+(A_8A_9T_1)a_2+\Sigma(A_7T_{j-3}+A_8T^1_{j-3}+A_9T^2_{j-3})a_j$$
$$+A_{11}b_1+A_{11}T_1b_2+\Sigma(A_{10}T_{j-3}+A_{11}T^2_{j-3})b_j+A_{12}c_2$$
$$+\Sigma A_{12}T^1_{j-3}c_j=0 \tag{17b}$$

$$A_{14}a_1+(A_{13}+A_{14}T_1)a_2+\Sigma(A_{13}T^1_{j-3}+A_{14}T^2_{j-3})a_j$$
$$+A_{15}b_2+\Sigma A_{15}T^1_{j-3}b_jA_{18}c_1+(A_{17}A_{18}T_1)c_2$$
$$+\Sigma(A_{16}T_{j-3}+A_{17}T^1_{j-3}+A_{18}T^2_{j-3})c_j=0 \tag{17c}$$

The satisfaction of Eqs. (17) at $n-2$ points given by the formula

$$X=X_m=\cos[(2m+1)/(n-2)\cdot\pi/2]$$
$$m=0,\,1,\,2,\,\ldots,\,m-3 \tag{18}$$

gives $3n-6$ equations in $3n$ unknown constants. The remaining six equations are obtained by the edge conditions. The zeros of the $3n\times 3n$ determinant obtained from the coefficient matrix of the $3n$ equations give frequencies for the normal mode of vibration. For determining the normalized mode shapes, any $3-1$ out of $3n$ equations are solved by taking one of the unknown constants as unity and the results are then normalized.

The determinant and the simultaneous equations are solved by Gauss elimination method making partial pivoting.

Edge Conditions

The following edge conditions are considered:
1) Clamped at both edges $X=-1$ and $X=1$ (C-C).
2) Free at the edge $X=-1$ and clamped at the edge $X=1$ (F-C). For clamped edge conditions: $B_1=B_2=W=0$. For free edge conditions: $M_X=N_X=Q_X=0$.

Table 2 Ω for monocoque C-C plate of varying thickness ($\alpha=3.0$ deg, $H=0.04$)

Mode of vibration	DMP	RMP $H_0=0.03$ $H_2=0.01$	PD	$H_0=0.02$ $H_2=0.02$	PD	$H_0=0.01$ $H_2=0.03$	PD
Fundamental	1.26509	1.25444	0.84	1.25223	1.02	1.25412	0.87
Second	3.14729	3.10139	1.46	3.08985	1.82	3.09547	1.65
Third	5.55005	5.44274	1.93	5.41573	2.42	5.42819	2.20
Fourth	8.26545	8.07614	2.29	8.03054	2.84	8.05381	2.56

Table 3 Ω for monocoque F-C plate of varying thickness ($\alpha=3.0$ deg, $H=0.04$)

Mode of vibration	DMP	RMP $H_0=0.03$ $H_2=0.01$	PD	$H_0=0.02$ $H_2=0.02$	PD	$H_0=0.01$ $H_2=0.03$	PD
Fundamental	0.34440	0.34435	0.01	0.34433	0.02	0.34435	0.01
Second	1.46907	1.46176	0.50	1.45863	0.71	1.45832	0.73
Third	3.40531	3.36957	1.05	3.35553	1.46	3.35546	1.46
Fourth	5.85137	5.76040	1.55	5.72837	2.10	5.73158	2.05

Numerical Results and Discussions

The frequency parameter Ω and normalized transverse deflection are computed for the first four normal modes of vibration for various values of α, H_0, H_f, and R_e. Ω is also computed by varying α and R_ρ in such a way that the total weight of the plate remains constant. The weight of the plate per unit length is $[\rho_1(2h_0+a\tan\alpha)+2\rho_2h_2]\,ag$, where g is the acceleration due to gravity. It is made nondimensional by dividing by ρ_1ga^2 and denoted by V. Therefore, $V=2H_0+\tan\alpha+2R_\rho H_2$. Unless otherwise mentioned, the values of the material constants are taken as $\nu_1=0.1$, $R_\nu=3.0$, $R_e=2000.0$, and $R_\rho=30.0$. K is taken equal to one because Yu[13] has shown that it is approximately equal to one for ordinary sandwiches and Koplik and Yu[14] have used the same value for sandwich spherical caps. The numerical results are computed by taking $n=20$. Higher values of n do not improve the results at least up to five decimal places.

To check the accuracy of the Chebyshev collocation method (CCM), the frequencies of sandwich plate of constant thickness computed 1) from the analysis of sandwich plate of variable thickness by taking $\alpha=0$ (CCM) and 2) from the analytic solution of sandwich plate of constant thickness (AS) are compared in Table 1. The results match perfectly up to five decimal places.

To check the validity of assumptions for displacements of sandwich plate of varying thickness taken in Eq. (1), the frequencies of the monocoque plate of varying thickness computed 1) by reducing the sandwich plate to monocoque plate by taking the core and the faces of the same material (RMP), and 2) directly from the analysis of monocoque plate (DMP) are compared and percentage difference (PD) is given in Table 2 for the C-C plate and in Table 3 for the F-C plate. The inital thickness of the monocoque plate is denoted by H. In the case of the RMP, $H = H_0 + H_2$.

The difference in the frequencies is due to the round off errors because a determinant of order 60 is computed in case of RMP and a determinant of order 40 is computed in case of DMP.

It is clear that the difference in the F-C case is less than that of the C-C case.

For the first four normal modes of vibration, Ω vs α, H_0, H_f, and R_e are plotted in Figs. 2–5, respectively. Ω vs α, when the variation in α is adjusted by H_0 or H_f in such a way that the total weight of the plate remains constant, is plotted in Fig. 6 for the C-C plate and in Fig. 7 for the F-C plate. Ω vs R_e, when the variation in R_e is adjusted by H_f, is plotted in Fig. 8. Normalized transverse deflection for the first four modes of vibration is plotted in Fig. 9 for the C-C plate and in Fig. 10 for the F-C plate. α is taken in degrees everywhere.

Figure 2 shows that Ω increases with the increase in α in all of the four modes. The rate of increase is almost the same in all of the four modes. The modewise rate of increase is higher in the F-C plate than in the C-C plate. Figure 3 shows that Ω increases with the increase in H_0 in all of the four modes. The rate of increase increases as one goes to higher modes and this increase is higher in the F-C plate than in the C-C plate. Figure 4 shows that, as H_f increases, Ω first increases up to a certain value of H_f and then increases in all of the four modes. The point from which the increase in Ω starts shifts toward the left as one goes to higher modes. This point comes earlier in the

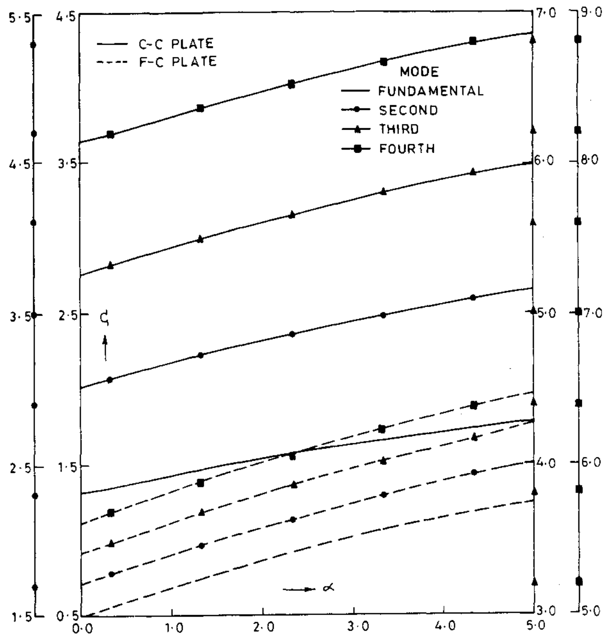

Fig. 2 Ω **vs** α **when** $H_0 = 0.02$, $H_f = 0.01$.

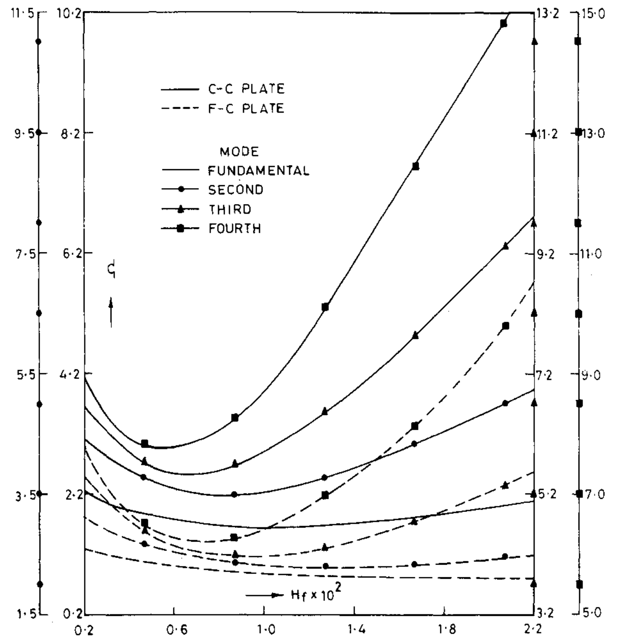

Fig. 4 Ω **vs** H_f **when** $\alpha = 2.0$, $H_0 = 0.04$.

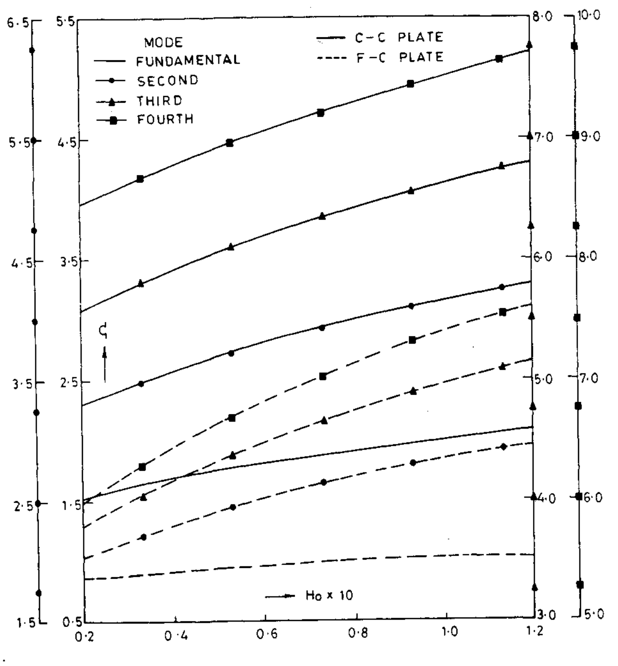

Fig. 3 Ω **vs** H_0 **when** $\alpha = 2.0$, $H_f = 0.01$.

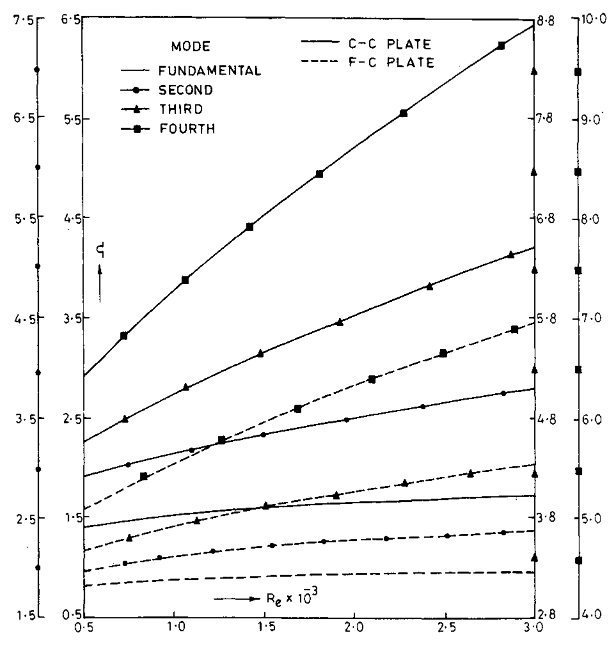

Fig. 5 Ω **vs** R_e **when** $\alpha = 2.5$, $H_0 = 0.03$, **and** $H_f = 0.01$.

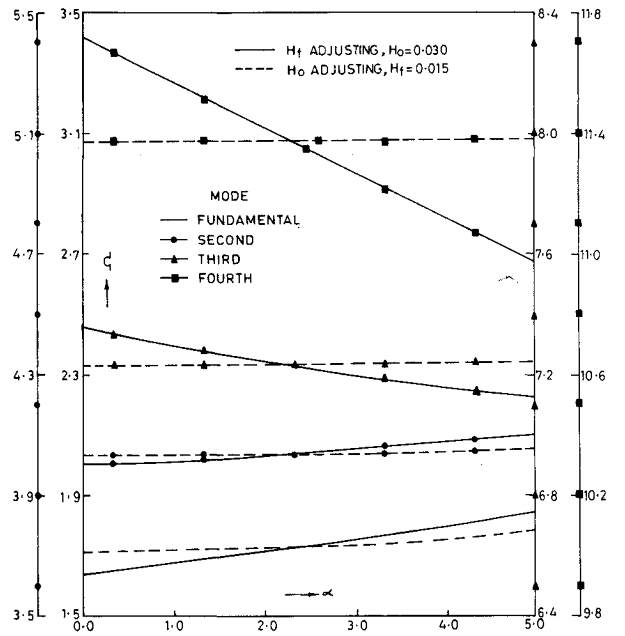

Fig. 6 Ω vs α for C-C plate when $V=1.0$.

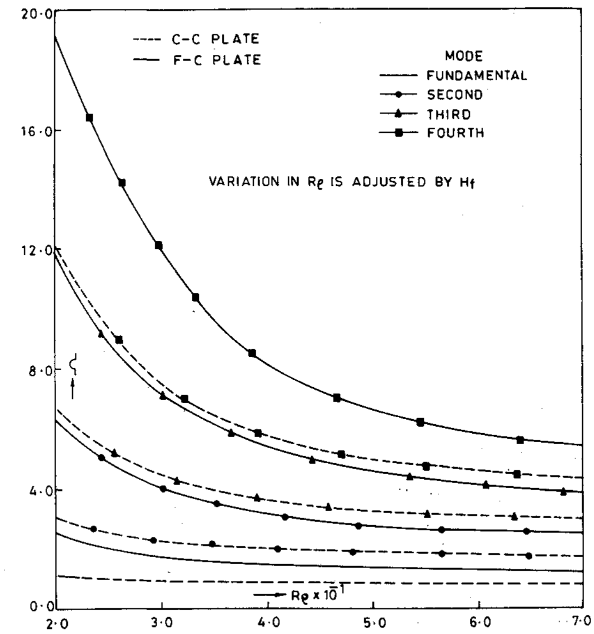

Fig. 8 Ω vs R_e when $\alpha=3.0$, $H_0=0.03$, and $V=1.0$.

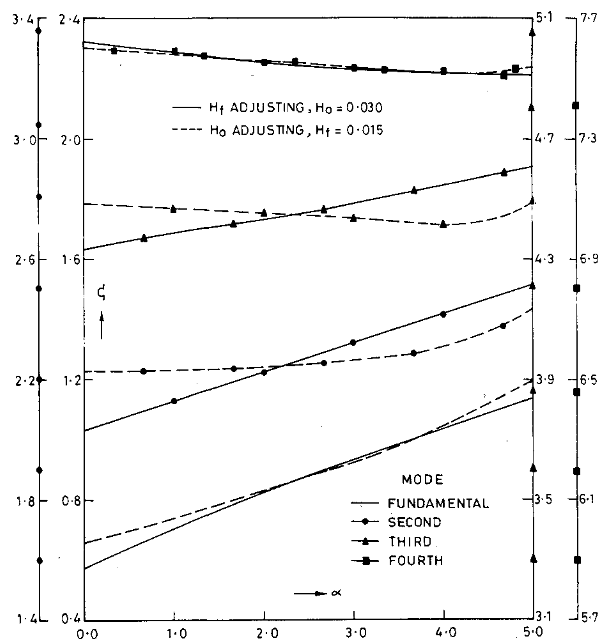

Fig. 7 Ω vs α for F-C plate when $V=1.0$.

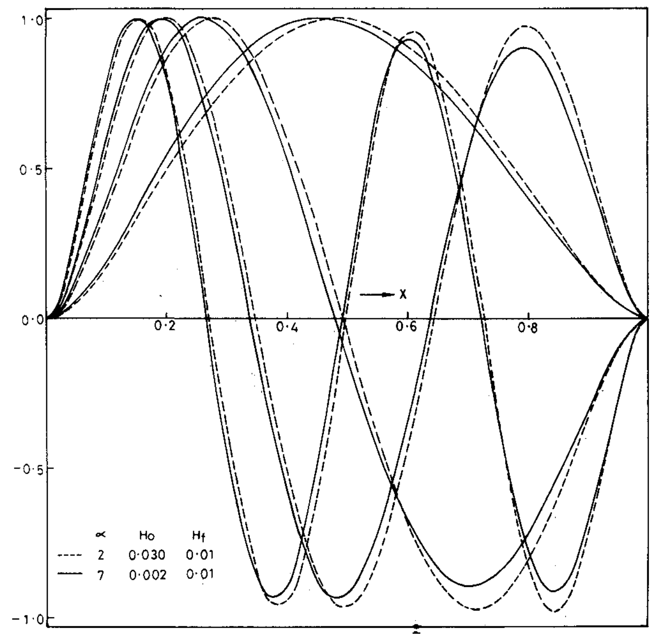

Fig. 9 Normalized transverse deflections vs X for C-C plate.

C-C plate than in the F-C plate. Figure 5 shows that Ω increases with the increase in R_e in all of the four modes. The rate of increase increases as one goes to higher modes. The rate of increase is higher in the C-C plate than in the F-C plate.

Figure 6 shows that in the C-C plate, when the increase in α is adjusted by H_f, Ω increases in the first and second modes but decreases in the third and fourth modes. The rate of increase decreases in the second mode. The rate of decrease increases in the fourth mode. Ω is not sensitive to α when the change in α is adjusted by H_0. Figure 7 shows that in the F-C plate, when the increase in α is adjusted by H_f, Ω increases in the first three modes but decreases in the fourth mode. The rate of increases decreases as one goes to higher modes so that Ω decreases in the fourth mode. When the change in α is adjusted by H_0, Ω increases in the first and second modes. The rate of increase increases for higher values of α. The rate of increase is very small in the beginning in the second mode. In the third and fourth modes, Ω decreases slightly in the beginning and then starts increasing. Figure 8 shows that, when the increase in R_e is adjusted by H_f, Ω decreases first rapidly and then slowly in all of the four modes. The rate of decrease increases as one goes to higher modes. The rate of decrease is higher in the C-C plate than in the F-C plate.

Figure 9 shows that in the C-C plate, the nodal lines and the points of extreme deflection shift toward the left as α increases. The magnitude of extreme deflection decreases with the increase in α. Figure 10 shows that in the F-C plate, the maximum deflection does not necessarily occur on the tip. As α increases, nodal lines in the second mode shift toward the right but in the third and fourth mode shift toward the left.

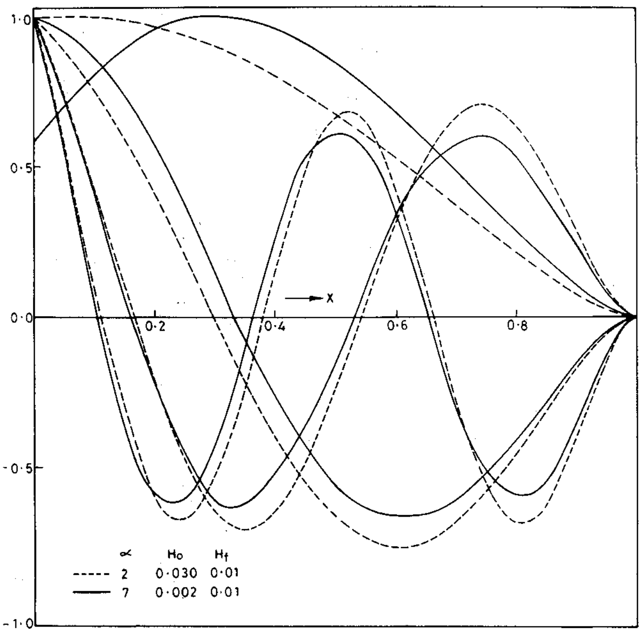

Fig. 10 Normalized transverse deflections vs X for F-C plate.

The points of extreme deflection shift toward the left in the second and higher modes. The magnitude of extreme deflection decreases with the increase in α.

Concluding Remarks

Some interesting things noted in the vibration of sandwich plates are as follows:

1) As H_f increases, the stiffness of the plate increases but the frequency decreases in the beginning and then increases from and after a certain value of H_f.

2) When α increases in such a way that the total weight of the plate remains constant, Ω increases in lower modes but decreases in higher modes.

3) Ω decreases in all of the modes of vibration when the increase in R_e is adjusted by H in such a way that the total weight of the plate remains constant.

4) In cantilever plates, the maximum deflections do not necessarily occur on the tip of the plate.

References

[1]Habip, L. M., "A Survey of Modern Developments in the Analysis of Sandwich Structures," *Applied Mechanics Review,* Vol. 18, No. 2, 1965, pp. 93–98.

[2]Plantema, F. J., *Sandwich Construction,* 1st ed., Wiley, New York, 1966.

[3]Bert, C. W., "Recent Research in Composite and Sandwich Plate Dynamics," *Shock and Vibration Digest,* Vol. 11, No. 10, 1979, pp. 13–23.

[4]Gupta, A. P., and Jain, M., "Axisymmetric Vibrations of Annular Sandwich Plates of Linearly Varying Thickness," *Journal of Sound and Vibration,* Vol. 80, No. 3, 1982, pp. 329–337.

[5]Paydar, N., and Adams, R., "Transverse Vibration of Annular Sandwich Plates with Linear Thickness Variation," *Journal of Sound and Vibration,* Vol. 131, No. 2, 1989, pp. 259–263.

[6]Taber, L. A., and Viano, D. C., "Comparison of Analytical and Experimental Results for Free Vibration of Non-Uniform Composite Beams," *Journal of Sound and Vibration,* Vol. 83, No. 2, 1982, pp. 219–228.

[7]Paydar, N., and Libove, C., "Stress Analysis of Sandwich Plates with Unidirectional Thickness Variation," *Journal of Applied Mechanics,* Vol. 53, No. 3, 1986, pp. 609–613.

[8]Paydar, N., and Libove, C., "Bending of Sandwich Plates of Variable Thickness," *Journal of Applied Mechanics,* Vol. 55, No. 2, 1988, pp. 419–424.

[9]Paydar, N., "Stress Analysis of Annular Sandwich Plates of Linearly Varying Thickness," *International Journal of Solids and Structures,* " Vol. 24, No. 3, 1988, pp. 313–320.

[10]Ko, C. L., "Dynamic Analysis for Free Vibrations of Rotating Sandwich Tapered Beams," *AIAA Journal,* Vol. 27, No. 10, 1989, pp. 1425–1433.

[11]Kar, R. C., and Sujata, T., "Dynamic Stability of a Tapered Symmetric Sandwich Beam," *Computers and Structures,* Vol. 40, No. 6, 1991, pp. 1441–1449.

[12]Yu, Y. Y., "A New Theory of Elastic Sandwich Plates—One Dimensional Case," *Journal of Applied Mechanics,* Vol. 26, No. 3, 1959, pp. 415–421.

[13]Yu, Y. Y., "Simple Thickness Shear Modes of Vibration of Infinite Sandwich Plates," *Journal of Applied Mechanics,* Vol. 26, No. 4, 1959, pp. 679–681.

[14]Koplik, B., and Yu, Y. Y., "Axisymmetric Vibrations of Homogeneous and Sandwich Spherical Caps," *Journal of Applied Mechanics,* Vol. 34, No. 3, 1967, pp. 667–673.